EE C222/ME C237 - Spring'18 - Lecture 12 Notes[1]

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Linear Time-Varying Systems

Khalil Section 4.6, Sastry Section 5.7

$$\dot{x} = A(t)x \qquad x(t) = \Phi(t,t_0)x(t_0) \tag{1}$$

- The state transition matrix $\Phi(t,t_0)$ satisfies the equations:

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0) \tag{2}$$

$$\frac{\partial}{\partial t_0}\Phi(t,t_0) = -\Phi(t,t_0)A(t_0) \tag{3}$$

- No eigenvalue test for stability in the time-varying case:

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix}$$

eigenvalues: $-0.25 \mp i0.25\sqrt{7}$ for all t, but unstable:

$$\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ e^{-0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

- For linear systems uniform asymptotic stability is equivalent to uniform exponential stability:

Theorem[2]: $x = 0$ is uniformly asymptotically stable if and only if

$$\|\Phi(t,t_0)\| \le ke^{-\lambda(t-t_0)} \text{ for some } k>0,\ \lambda>0.$$

- Last lecture: $V(t,x) = x^T P(t)x$ proves uniform exp. stability if

$$\begin{array}{rl} \text{(i)} & \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \\ \text{(ii)} & 0 < k_1 I \le P(t) \le k_2 I \\ \text{(iii)} & 0 < k_3 I \le Q(t) \text{ for all } t. \end{array}$$

The converse is also true:

Theorem: Suppose $x = 0$ is uniformly exponentially stable, $A(t)$ is continuous and bounded, $Q(t)$ is continuous and symmetric, and there exist $k_3, k_4 > 0$ such that

$$0 < k_3 I \le Q(t) \le k_4 I \text{ for all } t.$$

Then, there exists a symmetric $P(t)$ satisfying (i)–(ii) above.

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[2] Khalil Thm. 4.11, Sastry Thm. 5.33

Proof:

Time-invariant: $P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau$

Time-varying: $P(t) = \int_t^\infty \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) d\tau$

Using the Leibniz rule, property (3), and $\Phi(t, t) = I$ we obtain:

$$\begin{aligned}
\dot{P}(t) = &\int_t^\infty \left(\frac{\partial}{\partial t}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) + \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t) \right) d\tau \\
&\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\
= &\int_t^\infty \left(-A^T(t)\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) - \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)A(t) \right) d\tau \\
&\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\
= &-A^T(t)P(t) - P(t)A(t) - Q(t).
\end{aligned}$$

Lyapunov-based Feedback Design Examples

Model Reference Adaptive Control

Illustrated on a first order system:

$$\dot{y} = a^* y + u \tag{4}$$

where a^* is unknown.

Reference model:

$$\dot{y}_m = -a y_m + r(t) \quad a > 0,\ r(t) : \text{reference signal.} \tag{5}$$

Goal: Design a controller that guarantees $y(t) - y_m(t) \to 0$ without the knowledge of a^*.

If we knew a^*, we would choose:

$$u = -\underbrace{(a^* + a)}_{=:k^*} y + r(t) \quad \Rightarrow \quad \dot{y} = -ay + r(t).$$

The tracking error $\boxed{e(t) := y(t) - y_m(t)}$ satisfies:

$$\dot{e} = -ae \;\Rightarrow\; e(t) \to 0 \text{ exponentially.}$$

Adaptive design when a^* (therefore, k^*) is unknown:

$$u = -k(t)y + r(t)$$

where $\dot{k}(t)$ is to be designed. Then: $\quad \dot{e} = -ae - \underbrace{(k(t) - k^*)}_{=:\tilde{k}(t)} y.$

Use the Lyapunov function: $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$:

$$\begin{aligned}\dot{V} &= -ae^2 - \tilde{k}ey + \tilde{k}\dot{\tilde{k}} \\ &= -ae^2 + \tilde{k}(\dot{\tilde{k}} - ey).\end{aligned}$$

Note $\dot{\tilde{k}} = \dot{k}$ and choose $\boxed{\dot{k} = ey}$ so that $\dot{V} = -ae^2$.

This guarantees stability of $(e, \tilde{k}) = (0, 0)$ and boundedness of $(e(t), \tilde{k}(t))$ since the level sets of $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$ are positively invariant. In addition, if $r(t)$ bounded, then $y_m(t)$ in (5) is bounded, and so is $y(t) = y_m(t) + e(t)$. Then we can apply the Theorem from Lecture 11, page 3, to the time-varying model

$$\dot{e} = -ae - y(t)\tilde{k}, \quad \dot{\tilde{k}} = y(t)e,$$

and conclude from $\dot{V} = -ae^2$ that $e(t) \to 0$.

Whether $\tilde{k}(t) \to 0$ ($k(t) \to k^*$) depends on further properties of the reference signal $r(\cdot)$ that are beyond the scope of this lecture.

Backstepping

Khalil (Sec. 14.3), Sastry (Sec. 6.8)

Feedback stabilization: Given the system

$$\dot{x} = f(x) + g(x)u \tag{6}$$

with input u, design a control law $u = \alpha(x)$ such that $x = 0$ is asymptotically stable for the closed-loop system:

$$\dot{x} = f(x) + g(x)\alpha(x).$$

Backstepping is a technique that simplifies this task for a class of systems.

Suppose a stabilizing feedback $u = \alpha(X)$ is available for:

$$\dot{X} = F(X) + G(X)u \quad X \in \mathbb{R}^n, u \in \mathbb{R}$$

and suppose the closed-loop system admits a Lyapunov function $V(X)$ such that

$$\frac{\partial V}{\partial X}\Big(F(X) + G(X)\alpha(X)\Big) \leq -W(X) < 0 \quad \forall X \neq 0.$$

Can we modify $\alpha(X)$ to stabilize the augmented system below?

$$\boxed{\begin{aligned}\dot{X} &= F(X) + G(X)x \\ \dot{x} &= u.\end{aligned}}$$

Define the error variable $\boxed{z = x - \alpha(X)}$ and change variables:

$(X, x) \rightarrow (X, z)$:

$$\begin{aligned}\dot{X} &= F(X) + G(X)\alpha(X) + G(X)z \\ \dot{z} &= u - \dot{\alpha}(X, z)\end{aligned}$$

where $\dot{\alpha}(X,z) = \frac{\partial \alpha}{\partial X}\Big(F(X) + G(X)\alpha(X) + G(X)z\Big)$. Take the new Lyapunov function:

$$V_+(X,z) = V(X) + \frac{1}{2}z^2.$$

$$\dot{V}_+ = \underbrace{\frac{\partial V}{\partial X}\Big(F(X) + G(X)\alpha(X)\Big)}_{\leq -W(X)} + \underbrace{\frac{\partial V}{\partial X}G(X)z + z(u - \dot{\alpha})}_{= z\left(u - \dot{\alpha} + \frac{\partial V}{\partial X}G(X)\right)}$$

Let: $\boxed{u = \dot{\alpha} - \frac{\partial V}{\partial X}G(X) - kz, \quad k > 0.}$

Then, $\dot{V}_+ \leq -W(X) - kz^2 \;\Rightarrow\; (X, z) = 0$ is asymptotically stable.

Example:

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u.\end{aligned} \tag{7}$$

Treat x_2 as "virtual" control input for the x_1-subsystem:

$$\begin{aligned}\alpha(x_1) &= -k_1x_1 - x_1^2 \quad k_1 > 0 \\ V_1(x_1) &= \frac{1}{2}x_1^2.\end{aligned}$$

Apply backstepping:

$$\begin{aligned}z_2 &= x_2 - \alpha(x_1) = x_2 + k_1x_1 + x_1^2 \\ \dot{z}_2 &= u - \dot{\alpha} \\ u &= \dot{\alpha} - \frac{\partial V_1}{\partial x_1} - k_2z_2, \quad k_2 > 0 \\ &= \underbrace{-(k_1 + 2x_1)(x_1^2 + x_2)}_{= \dot{\alpha}} - \underbrace{x_1}_{= \frac{\partial V_1}{\partial x_1}} - k_2\underbrace{(x_2 + k_1x_1 + x_1^2)}_{= z_2}.\end{aligned}$$